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On a Problem in Monotone Approximation

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Let $f \in C[-1, 1]$. A sufficient condition is given which ensures that the nth polynomial of best approximation to f is increasing for n sufficiently large. Using this condition, we are able to give a counterexample to a theorem announced by Tzimbalario [6].

1. INTRODUCTION

Let n, k be nonnegative integers and let Π_n denote the set of algebraic polynomials of degree n or less. Let $C^k[-1, 1]$ denote the class of functions which have a continuous kth derivative on $[-1, 1]$($C[-1, 1]$ will mean $C^0[-1, 1]$).

For $f \in C[-1, 1]$, define

$$E_n(f) = \min_{p \in \Pi_n} \|p - f\|,$$

where $\| \quad \|$ denotes the uniform norm on $[-1, 1]$. It is well known that for each n the above minimum is attained by a unique element in Π_n. We call this element the nth algebraic polynomial of best approximation to f. Later, if no confusion is likely to occur, we will always denote it by p_n for any given f.

The problem we study in this paper is the following: Let $f \in C[-1, 1]$, and assume that there exists a $\delta > 0$, such that

$$(f(x_1) - f(x_2))/(x_1 - x_2) \geqslant \delta \tag{1.1}$$

for all $x_1, x_2 \in [-1, 1]$ with $x_1 \neq x_2$. What extra condition on f is needed to ensure that p_n is increasing for all n sufficiently large?

Roulier [4] showed that $f \in C^2[-1, 1]$ is such a condition. Also in [4], Roulier asked: if $f \in C^1[-1, 1]$ and satisfies (1.1) (or equivalently $f'(x) \geqslant \delta$

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for $x \in [-1, 1]$), is p_n increasing for n sufficiently large? In [5], Roulier conjectured that the answer is negative.

In answering this question, Tzimbalario [6] announced the following theorem:

THEOREM 1.1. *Let f be a continuous function on $[-1, 1]$ with f' not in some* Lip α, $\alpha < 1$, *and $f' \geqslant \delta$ for some strictly positive δ. Then there are infinitely many n for which p_n is not increasing.*

Fletcher and Roulier discussed this problem in [2]. Their main results there are the following two theorems.

THEOREM 1.2. *Let α be given in the interval $0 < \alpha < 1$. There exists $f \in C^1[-1, 1]$ for which*

$$f'(x) \geqslant \delta > 0, \qquad x \in [-1, 1] \tag{1.2}$$

and

$$f' \in \operatorname{Lip} \alpha, \qquad \text{but} \qquad f' \notin \operatorname{Lip}(\alpha + \varepsilon) \tag{1.3}$$

for any $\varepsilon > 0$, such that there are infinitely many n for which p_n is not increasing on $[-1, 1]$.

THEOREM 1.3. *Let $0 < \alpha < 1$ be given. There exists a function $f \in C^1[-1, 1]$ for which* (1.2) *and* (1.3) *hold and such that p_n is icreasing for all n sufficiently large.*

Also in [2], Fletcher and Roulier drew the conclusion that Theorem 1.3 provides counterexamples to Tzimbalario's Theorem (Theorem 1.1).

We have noted that there might be the following different interpretations of Tzimbalario's Theorem:

THEOREM 1.1a. *Let f be a function in $C^1[-1, 1]$ for which* (1.2) *holds. If there exists α in the interval* (0, 1), *such that $f' \notin$* Lip α, *then there are infinitely many n for which p_n is not increasing.*

THEOREM 1.1b. *Let f be a function in $C^1[-1, 1]$ for which* (1.2) *holds. If $f' \notin \bigcup_{0 < \alpha < 1}$* Lip α, *then there are infinitely many n for which p_n is not increasing.*

Theorem 1.3 only provides counterexamples to Theorem 1.1a, and it is obvious that a counterexample to Theorem 1.1b will automatically be one to Theorem 1.1a.

The purpose of this paper is to prove a stronger result on the positive aspect of the problem, and to provide a counterexample to Theorem 1.1b.

2. MAIN RESULTS

THEOREM 2.1. *Let f be a function in $C[-1,1]$, satisfying (1.1). If $E_n(f)=o(n^{-2})$, then p_n is increasing for all n sufficiently large.*

Since $f\in C^2[-1,1]$ necessarily means $E_n(f)=o(n^{-2})$, by Jackson's Theorem [1, pp. 147], Theorem 2.1 is stronger than Roulier's result [3].

THEOREM 2.2. *There exists $f\in C^1[-1,1]$ for which (1.2) holds, but $f'\notin\bigcup_{0<\alpha<1}\operatorname{Lip}\alpha$, such that p_n is increasing on $[-1,1]$ when n is sufficiently large.*

The contradiction between Theorems 2.2 and 1.1 is apparent.

LEMMA. *Let $f\in C^1[-1,1]$ and $q_n\in\Pi_n$. If $\|q_n-f\|=o(n^{-2})$ then $\|q_n''\|=o(n^2)$.*

Proof. Let n be fixed, and choose k so that $2^k<n\leqslant 2^{k+1}$. Write

$$q_n=(q_n-q_{2^{k+1}})+\sum_{i=0}^{k}(q_{2^{i+1}}-q_{2^i})+q_1.$$

Since $q_1''=0$, $\|q_n''\|\leqslant\|q_n''-q_{2^{k+1}}''\|+\sum_{i=0}^{k}\|q_{2^{i+1}}''-q_{2^i}''\|$. Since $\|q_n-q_{2^{k+1}}\|\leqslant\|q_n-f\|+\|f-q_{2^{k+1}}\|=o(n^{-2})$, by Markov's Inequality, $\|q_n''-q_{2^{k+1}}''\|=o(n^2)$.

Let $A(n)=\sum_{i=0}^{k}\|q_{2^{i+1}}''-q_{2^i}''\|$. It remains to show that $A(n)=o(n^2)$, i.e., that for any given $\varepsilon>0$, $A(n)<\varepsilon n^2$ for n sufficiently large.

We introduce some new notations by letting $v_i=q_{2^i}$ and $\beta_i=\sup_{j\geqslant i}\|v_j-f\|$. We have $\|v_{i+1}-v_i\|\leqslant\beta_{i+1}+\beta_i\leqslant 2\beta_i$.

By Markov's Inequality, $\|v_{i+1}''-v_i''\|\leqslant 2\beta_i(2^{i+1})^4$. As $\|q_n-f\|=o(n^{-2})$, we may assume that $\beta_i\leqslant\alpha_i(2^i)^{-2}$ where $\alpha_i\downarrow 0$.

Now we have

$$A(n)\leqslant\sum_{i=0}^{k}2\beta_i(2^{i+1})^4\leqslant\sum_{i=0}^{k}2\alpha_i(2^i)^{-2}(2^{i+1})^4=32\sum_{i=0}^{k}\alpha_i 4^i.$$

Given $\varepsilon>0$, select m so that $\alpha_i<\varepsilon$ when $i\geqslant m$. Select $N\geqslant m$ so that $n^{-2}\sum_{i=0}^{m-1}\alpha_i 4^i<\varepsilon$ when $n\geqslant N$. Then for any $n\geqslant N$ we will have

$$\begin{aligned}(1/32)n^{-2}A(n)&\leqslant n^{-2}\sum_{i=0}^{m-1}\alpha_i 4^i+n^{-2}\sum_{i=m}^{k}\alpha_i 4^i\\&\leqslant\varepsilon+n^{-2}\varepsilon\sum_{i=m}^{k}4^i\\&\leqslant\varepsilon+n^{-2}\varepsilon 4^{k+1}\leqslant 5\varepsilon.\end{aligned}$$

Proof of Theorem 2.1. Let δ be as in (1.1); we will show that $p'(x) \geqslant \delta/4$ for n sufficiently large. Suppose not; then there is an infinite subset of natural numbers N^* such that the following is true for $n \in N^*$,

$$p'_n(x_n) < \delta/4, \tag{2.1}$$

where x_n, $n \in N^*$, is a sequence of points in the interval $[-1, 1]$. By the Mean-Value Theorem and the lemma we have just proved, we have, for n sufficiently large, that

$$|p'_n(x_n) - p'_n(x_n \pm h)| = |p''_n(\xi)|\, h \leqslant \|p''_n\|\, n^{-2} < \delta/4, \tag{2.2}$$

where $0 \leqslant h \leqslant n^{-2}$ and the sign $+$ or $-$ is chosen so that $x_n + h$ or $x_n - h$ is in the interval $[-1, 1]$. In the following, for the convenience of writing, we assume that $+$ has always been chosen.

By (2.1) and (2.2)

$$\begin{aligned} p'_n(x_n + h) &= p'_n(x_n + h) - p'_n(x_n) + p'_n(x_n) \\ &< \delta/4 + \delta/4 = \delta/2. \end{aligned}$$

Using the Mean-Value Theorem again, we have

$$p_n(x_n + n^{-2}) - p_n(x_n) < \delta/(2n^2). \tag{2.3}$$

As $\|p_n - f\| = o(n^{-2})$ we may assume that $\|f - p_n\| < \delta/(4n^2)$. Using this last inequality and (2.3), we get

$$\begin{aligned} &f(x_n + n^{-2}) - f(x_n) \\ &\quad = [f(x_n + n^{-2}) - p_n(x_n + n^{-2})] \\ &\qquad + [p_n(x_n + n^{-2}) - p_n(x_n)] + [p_n(x_n) - f(x_n)] \\ &\quad < \delta/(4n^2) + \delta/(2n^2) + \delta/(4n^2) = \delta/n^2. \end{aligned}$$

This contradicts the assumption (1.1), and completes the proof.

Proof of Theorem 2.2. We choose the basic interval here to be $[0, 1]$ instead of $[-1, 1]$; there is no loss of generality in doing this.

Let

$$g(x) = \begin{cases} x/\ln(2/x), & x \in (0, 1] \\ 0, & x = 0. \end{cases}$$

Then

$$g'(x) = \begin{cases} (\ln(2/x) + 1)/\ln(2/x)^2, & x \in (0, 1] \\ 0, & x = 0. \end{cases}$$

Let $f(x) = g(x) + \delta x$, where δ is as in (1.1). It is obvious that f satisfies (1.2). Since

$$\lim_{x \to 0} \frac{g'(x)}{x^{\alpha}} = \infty$$

for any $0 < \alpha < 1$, we infer

$$g'(x) \notin \text{Lip}\, \alpha \qquad \text{for every} \quad \alpha \quad \text{satisfying} \quad 0 < \alpha < 1.$$

By Theorem 2.1, the proof will be completed if we can show that $E_n(f) = o(n^{-2})$. As $E_n(f) = E_n(g)$ for $n \geq 1$, it suffices to show that $E_n(g) = o(n^{-2})$. Consider

$$G(x) = \begin{cases} x^2/\ln(2/x^2), & x \in [-1, 0) \cup (0, 1] \\ 0, & x = 0. \end{cases}$$

Differentiating $G(x)$ twice, we observe that $G \in C^2[-1, 1]$. By Jackson's Theorem $E_n(G) = o(n^{-2})$. Let Q_{2n} be the $2n$th best approximation polynomial of G. As G is even, so is Q_{2n} [3, Chapt. 2, Problem 3], and therefore $Q_{2n}(x) = q_n(x^2)$ where q_n is a polynomial of degree n or less. We have now

$$\begin{aligned} \| g(x) - q_n(x) \|_{[0,1]} &= \left\| x \Big/ \ln \frac{2}{x} - q_n(x) \right\|_{[0,1]} \\ &= \left\| x^2 \Big/ \ln \frac{2}{x^2} - q_n(x^2) \right\|_{[-1,1]} \\ &= \| G(x) - Q_{2n}(x) \|_{[-1,1]} = o(n^{-2}). \end{aligned}$$

So we have $E_n(g) = o(n^{-2})$ and Theorem 2.2 is proved.

3. Comment and Conjecture

If we read carefully the proofs of Theorems 1.2 and 1.3 by Fletcher and Roulier [2], we find that the example in the proof of Theorem 1.3 satisfies the condition of Theorem 2.1, i.e., $E_n(f) = o(n^{-2})$, while the one in that of Theorem 1.2 does not. The result of Theorem 2.2 also exhibits the power of Theorem 2.1.

We make the following conjecture:

CONJECTURE. Theorem 2.1 cannot be improved, in the sense that there exists $f \in C[-1, 1]$, satisfying (1.1), and $E_n(f) = O(n^{-2})$, such that $p_n(x)$ is not increasing for infinitely many n.

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